HOMEWORK 3 SOLUTIONS

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Problem 5.1

Let $P(n)$ be the statement "$n^3 - n$ is divisible by 3."
Induction basis. Note that $1^3 - 1 = 0 (= 3 \cdot 0)$ is divisible by 3, hence $P(1)$ is true.
Induction hypothesis. Suppose that $P(k)$ is true for some integer $k \geq 1$.
Induction step. By the induction hypothesis, there exists an integer q with $k^3 - k = 3q$. Now

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3k^2 + 3k = 3q + 3k^2 + 3k = 3(q + k^2 + k),$$

so since $q + k^2 + k$ is an integer, $(k+1)^3 - (k+1)$ is divisible by 3, i.e. $P(k+1)$ is true. By induction we are done.

Problem 5.2

Let $P(n)$ be the statement "$n^3 \leq 2^n$".
Base case: $10^3 = 1000$ and $2^{10} = 1024$. Since $1000 \leq 1024$, we have that $P(10)$ is true.
Induction Hypothesis: Suppose that $P(m)$ is true for some integer $m \geq 10$.
Induction Step:

$$\begin{aligned}(m+1)^3 &= m^3 + 3m^2 + 3m + 1 \\ &\leq 2^m + 3m^2 + 3m + 1 \text{ by our induction hypothesis} \\ &= 2^m + 3m^2\left(1 + \frac{1}{m} + \frac{1}{3m^2}\right)\end{aligned}$$

Since $m \geq 10$, we know $\frac{1}{m} \leq 1$ and $\frac{1}{3m^2} \leq 1$. Thus,

$$1 + \frac{1}{m} + \frac{1}{3m^2} \leq 3.$$

Going back to our computation, this means

$$\begin{aligned}2^m + 3m^2\left(1 + \frac{1}{m} + \frac{1}{3m^2}\right) &\leq 2^m + 3m^2(3) \\ &= 2^m + 9m^2 \\ &\leq 2^m + m^3 \text{ since } 9 \leq m \\ &\leq 2^m + 2^m \text{ by our induction hypothesis} \\ &= 2^{m+1}\end{aligned}$$

Thus, $(m+1)^3 \leq 2^{m+1}$, so $P(m+1)$ is true. By induction, $n^3 \leq 2^n$ for all integers $n \geq 10$.

Problem 5.4

Let $P(n)$ be the statement

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}.$$

Base case: When $n = 0$ we have $\sum_{i=0}^{0} x^i = x^0 = 1$ and $\frac{1-x^{0+1}}{1-x} = \frac{1-x}{1-x} = 1$. Note: it's critical here that $x \neq 1$, otherwise we would be dividing by 0! Thus, $P(0)$ is true.
Induction Hypothesis: Suppose that $P(k)$ is true for some integer $k \geq 0$.
Induction Step:

$$\begin{aligned}
\sum_{i=0}^{k+1} x^i &= \sum_{i=0}^{k} x^i + x^{k+1} \\
&= \frac{1-x^{k+1}}{1-x} + x^{k+1} \text{ by our induction hypothesis} \\
&= \frac{1-x^{k+1}+(1-x)x^{k+1}}{1-x} \\
&= \frac{1-x^{(k+1)+1}}{1-x}.
\end{aligned}$$

Thus $P(k+1)$ is true. By induction, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$ for all integers $n \geq 0$, provided that $x \neq 1$.

Problem 5.6

Let $P(n)$ be the statement "$u_n = n3^{n-1}$".
Induction basis. Since $u_0 = 0$, which equals $0 \cdot 3^0$, $P(0)$ is true.
Induction hypothesis. Suppose that $P(k)$ is true for some integer $k \geq 0$, in other words that $u_k = k \cdot 3^{k-1}$.
Induction step. Using the recursion for $n = k$ we find

$$u_{k+1} = 3u_k + 3^k = 3 \cdot k \cdot 3^{k-1} + 3^k = k \cdot 3^k + 3^k = (k+1) \cdot 3^k = (k+1) \cdot 3^{(k+1)-1},$$

hence $P(k+1)$ is true. By induction, $P(n)$ is true for all integers n.

Problem 5.7

(i) Let $P(n)$ be the statement "$x^n y^n = (xy)^n$."
Base case: When $n = 0$ we have $x^0 y^0 = 1 \cdot 1 = 1$ and $(xy)^0 = 1$. Thus $P(0)$ is true.
Induction Hypothesis: Suppose that $x^k y^k = (xy)^k$ for some integer $k \geq 0$.
Induction Step:

$$\begin{aligned}
x^{k+1} y^{k+1} &= x \cdot x^k \cdot y \cdot y^k \\
&= (xy)(x^k y^k) \text{ since multiplication is commutative} \\
&= (xy)(xy)^k \text{ by our induction hypothesis} \\
&= (xy)^{k+1}.
\end{aligned}$$

Thus $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \geq 0$.

(ii) Let $P(n)$ be the statement "$x^{m+n} = x^m x^n$."
Base case: When $n = 0$, we have $x^{m+0} = x^m$ and $x^m x^0 = x^m \cdot 1 = x^m$. Thus $P(0)$ is true.
Induction Hypothesis: Suppose $P(k)$ is true for some integer $k \geq 0$.

Induction Step:

$$\begin{aligned} x^{m+k+1} &= x^{m+k} \cdot x \\ &= x^m x^k \cdot x \text{ by our induction hypothesis} \\ &= x^m x^{k+1}. \end{aligned}$$

Thus $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \geq 0$.

(iii) Let $P(n)$ be the statement "$(x^m)^n = x^{mn}$."
Base case: When $n = 0$ we have $(x^m)^0 = 1$ and $x^{m\cdot 0} = 1$. Thus $P(0)$ is true.
Induction Hypothesis: Suppose $P(k)$ is true for some integer $k \geq 0$.
Induction Step:

$$\begin{aligned} (x^m)^{k+1} &= (x^m)(x^m)^k \\ &= (x^m)(x^{mk}) \text{ by our induction hypothesis} \\ &= x^{m+mk} \text{ by part (ii)} \\ &= x^{m(k+1)} \end{aligned}$$

Thus $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \geq 0$.

Problem 6.4

$x \in A$	$x \in B$	$x \in C$	$x \in B \cup C$	$x \in A \cap (B \cup C)$	$x \in A \cap B$	$x \in A \cap C$	$x \in (A \cap B) \cup (A \cap C)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since columns 5 and 8 are identical we conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Problem 6.5

(i) We need to show two things: (1) $A \subseteq B \implies A \cup B = B$ and (2) $A \cup B = B \implies A \subseteq B$.

(1) Suppose $A \subseteq B$. If $x \in B$ then $x \in A$ or $x \in B$, so $B \subseteq A \cup B$. (Note: this is always true and didn't rely on the assumption $A \subseteq B$). Now suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then since $A \subseteq B$ we must have $x \in B$. Otherwise $x \in B$ to begin with. In either case, if $x \in A \cup B$ then $x \in B$, so $A \cup B \subseteq B$. Therefore $A \cup B = B$, so $A \subseteq B \implies A \cup B = B$.

(2) Suppose $A \cup B = B$. Let $x \in A$. Then $x \in A$ or $x \in B$, so $x \in A \cup B = A \cup B$, so we have $x \in B$. Thus $A \subseteq B$.

(ii) We need to show two things: (1) $A \subseteq B \implies A \cap B = A$ and (2) $A \cap B = A \implies A \subseteq B$.

(1) Suppose $A \subseteq B$. If $x \in A \cap B$ then $x \in A$ and $x \in B$, so in particular $x \in A$. Thus $A \cap B \subseteq A$. (Note: this is always true and didn't rely on the assumption $A \subseteq B$). Now suppose $x \in A$. Since $A \subseteq B$ we have $x \in B$. Thus $x \in A$ and $x \in B$, so $x \in A \cap B$, so $A \subset A \cap B$. Therefore $A \cap B = A$.

(2) Suppose $A \cap B = A$. Let $x \in A$. Since $A = A \cap B$, we know $x \in B$. Thus $A \subseteq B$.

Problem 6.6

We prove this by contradiction, so assume that $x \in A - C$. Then by definition $x \in A$ and $x \notin C$. Since $x \in A$ and it is given that $x \in B$ we find that $x \in A \cap B$. By $A \cap B \subseteq C$, we now also find that $x \in C$. However, this contradicts the fact that $x \notin C$. Therefore, our assumption is false and we conclude that $x \notin A - C$.

Problem 6.7

$A \subseteq B$ is equivalent to the statement "$x \in A \implies x \in B$." This is equivalent to its contrapositive "$x \notin B \implies x \in A$." Within a universal set, to say $x \notin A$ is to say $x \notin A^c$. In other words, the contrapositive is equivalent to "$x \in B^c \implies x \in A^c$." This is equivalent to saying $B^c \subseteq A^c$.